ON CERTAIN SOLUTIONS OF AN INTEGRAL EQUATION OF THE THEORY OF A LINEARLY DEFORMABLE FOUNDATION

(O NEKOTORYKH RESHENIIAKH INTEGRAL'NOGO URAVNENIIA TEORII LINEINO DEFORMIRUEMOGO OSNOVANIIA)

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An effective solution of the integral Equation

$$w(x,y)=\iint p(\xi,\eta)\,\mathbf{K}(r)\,d\xi\,d\eta \qquad \left(r=\sqrt{(x-\xi)^2+(y-\eta)^2}\right) \tag{0.1}$$

is constructed herein for an elliptic domain in the case of the power kernel $\mathbf{K}(r)=Cr^{-1-k}$. Equation (0.1) is hence related to the Dirichlet problem for a certain second order differential operator. Such a method permits finding the eigenfunctions of Equation (0.1) which are kindred to Lamé ellipsoidal functions and when $w(x,y)$ is a polynomial permits the calculation of $p(x,y)$ as well as the continuation of $w(x,y)$ outside the limits of the domain of integration. In passing, formulas are obtained for the evaluation of certain integrals analogous to potentials and other results are found which have analogs in Newton potential theory and Lamé ellipsoidal functions.

The connection between the settling $w(x,y)$ and the pressure $p(x,y)$ in the theory of a linearly deformable foundation is taken in the form of (0.1) where the kernel $\mathbf{K}(r)$ is a monotonely decreasing function, $\mathbf{K}(\infty)=0$, has a singularity of not higher than the second order at the point $r=0$ (in order to insure convergence of the integral).

No special assumptions are made in this theory relative to the nature of the elastic deformation of the underlying space or layer. For a homogeneous elastic half-space $\mathbf{K}(r)=Cr^{-1}$.

Other forms of the dependence $\mathbf{K}(r)$, discussed in [1], are possible for inhomogeneities and other deviations from the hypothesis of an elastic half-space. In particular, a kernel of the form $\mathbf{K}(r)=Cr^{-1-k}$ is mentioned. This latter case is usually interpreted as corresponding to the growth in the modulus of elasticity with depth in proportion to z^k $(k>0)$ However, such an interpretetion requires special assumptions [2].

Attention was turned in [3] to the fact that the kernel Cr^{-1-k}, where

$$r=\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}$$

is cancelled by the operator

$$\Delta^*=\Delta+kz^{-1}\frac{\partial}{\partial z} \tag{0.2}$$

The connection with the Dirichlet problem for this operator was used to solve (0.1) in the case of w = const An effective solution of (0.1) was given in [4] for the case of $w(x,y)$ · a polynomial, based on the use of contour integration. However, this method does not permit the calculation of the value of $w(x,y)$ outside the boundaries of the domain of integration.

Moreover, the theorem proved in [4] that $w(x,y)$ is a polynomial if $p(x,y)$ has the form of a polynomial multiplied by the expression

$$(1 - x^2/a^2 - y^2/b^2)^{1/2(k-1)}$$

is not accompanied by the inverse theorem and the applicability of the method of undetermined coefficients always when $w(x,y)$ is a polynomial thereby remains unproved.

This inadequacy is removed herein, where a proof of this inverse theorem is presented at the end of section 4.

1. The connection between Equation (0.1) and the Dirichlet problem broadens the possibility of an approach to the solution of this equation in all cases when the differential operator admits of separation of variables. However, not every kernel admits of such a possibility. Let us clarify when this will hold. Components of the Laplacian $\triangle$ and multiplication by the constant β enter naturally into the operator $\triangle^*$ which cancels the kernel $\mathbf{K}(r)$, $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}$ Non-positive sign of β, $\beta = -\gamma^2$ is required for $\mathbf{K}(r)$ to be monotone. Only $\partial/\partial z$ of the first derivatives may enter and, with the factor kz^{-1} besides, since $\partial \mathbf{K}/\partial z = zr^{-1}\mathbf{K}(r)$. Therefore, the operator has the form

$$\triangle^* = \triangle + \frac{k}{z}\frac{\partial}{\partial z} - \gamma^2 \tag{1.1}$$

and Equation

$$\mathbf{K}''(r) + \frac{2+k}{r}\mathbf{K}'(r) - \gamma^2 \mathbf{K}(r) = 0 \tag{1.2}$$

is obtained for $\mathbf{K}(r)$.

(The constant factor k may not be replaced by the function $\varphi(r)$ since the expression $\triangle^*$ does not contain the parameters ξ and η).

Then the Macdonald functions (with a supplementary power factor)

$$\mathbf{K}(r) = Cr^{-1/2-1/2k}K_{-1/2-1/2k}(\gamma r) \tag{1.3}$$

are possible kernels.

When $\gamma = 0$, the power function is $\mathbf{K}(r) = Cr^{-1-k}$. As regards the constant k, keeping in mind the singularity of the kernel at $r = 0$, we must have $k > -1$; for convergence of the integral $k < 1$ is necessary. Thus $-1 < k < 1$ for all γ.

In case $\gamma \neq 0$ the construction of normal solutions encounters considerable difficulties associated with the necessity for a numerical solution of transcendental equations (in the form of an infinite-order determinant) from which the characteristic values of the accessory parameters in the differential equations obtained in the separation will be found.

In the $\gamma = 0$ case this difficulty is absent and the theory is developed more-or-less analogously to the theory of Newtonian potential for an elliptical disk. The analysis of the corresponding Dirichlet problem for $\triangle^*$ in place of (0.1) broadens the sphere of application of the theory and makes possible the transfer of the results to other domains.

Let us consider Equation

$$\triangle^* v = \Delta v + \frac{k}{z}\frac{\partial v}{\partial z} = 0 \qquad (-1 < k < 1) \tag{1.4}$$

in the domain $z \geqslant 0$ and its solution in the form

$$v(x, y, z) = \iint \frac{f(\xi, \eta)\, d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta)^2 + z^2]^{1/2(1+k)}} \tag{1.5}$$

in place of (0.1).

We assume the function $f(\xi, \eta)$, the source density, to be continuous almost everywhere on the boundary $z = 0$ and to satisfy the Lipschitz condition.

It was proved in [3] that uner these assumptions

$$\lim_{z\to 0} z^k \frac{\partial v}{\partial z} = -2\pi f(x, y) \tag{1.6}$$

Requirements for uniqueness of the solution arise in connection with (1.5) and (1.6): it must be $O(R^{-1-k})$ at infinity and the function $z^k \partial v/\partial z$ should have a finite limit almost everywhere for $z = 0$. Under these requirements, the uniqueness is obtained by the customary method from the identity

$$\int_S \varphi z^k \frac{\partial \varphi}{\partial n} dS = \int_T \varphi z^k \triangle^* \varphi\, d\tau + \int_T z^k (\nabla\varphi)^2\, d\tau \tag{1.7}$$

applied to the spherical segment T

$$\geqslant h, \qquad x^2 + y^2 + z^2 \leqslant R^2 \qquad (h\to 0,\ R\to\infty)$$

Certain integrals of the form of (1.5) are evaluated in the next section by reducing them to single integrals. It is proved that if the function $f(x, y)$ has the form of the product of $t^{1/2(k-1)}$ and a polynomial $P(t)$, where $t = 1 - x^2/a^2 - y^2/b^2$, then $F(x, y) = V(x, y, 0)$ within a disk is a polynomial. Simultaneously formulas are obtained which represent $V(x, y, z)$ everywhere outside the disk by single integrals (containing parameters). This proposition is in itself an interesting analog of the known proposition on the Newtonian potential of an ellipsoid with an ellipsoidal distribution of the attracting masses. Application of this result to the Dirichlet problem is limited because of the dependences connecting the coefficients of the polynomial $F(x, y)$. Use of known pressure distributions in the solution of contact problems [5] and [6] corresponds to this in the theory of elasticity.

2. Let us use ellipsoidal coordinates which are roots of Equation

$$\Phi(s) \equiv 1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{s} \tag{2.1}$$

Here $\lambda > 0 > \mu > -b^2 > \nu > -a^2$. For brevity, let us introduce the functions

$$\Psi(s) = (a^2+s)(b^2+s)s, \quad \Psi(s)\Phi(s) = (s-\lambda)(s-\mu)(s-\nu) \tag{2.2}$$

Formulas for the transformation to Cartesian coordinates and Lamé coefficients are

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{a^2(a^2-b^2)}, \ldots \quad H_\lambda{}^2 = \frac{(\lambda-\mu)(\lambda-\nu)}{4\Psi(\lambda)}, \ldots \tag{2.3}$$

(The unwritten formulas are obtained by a circular permutation, where $c^2 = 0$). In these coordinates the Laplace operator is

$$\Delta V = \sum_{\lambda,\mu,\nu} \frac{4\Psi(\lambda)}{(\lambda-\mu)(\lambda-\nu)}\left\{\frac{\partial^2 V}{\partial\lambda^2} + \frac{1}{2}\frac{\Psi'(\lambda)}{\Psi(\lambda)}\frac{\partial V}{\partial\lambda}\right\} \tag{2.4}$$

while

$$\frac{\partial V}{\partial z} = 2z \sum_{\lambda,\mu,\nu} \frac{\Psi(\lambda)}{\lambda(\lambda-\mu)(\lambda-\nu)}\frac{\partial V}{\partial\lambda} \tag{2.5}$$

Therefore

$$\Delta^* V = \sum_{\lambda,\mu,\nu} \frac{4\Psi(\lambda)}{(\lambda-\mu)(\lambda-\nu)}\left\{\frac{\partial^2 V}{\partial\lambda^2} + \frac{1}{2}\left[\frac{\Psi'(\lambda)}{\Psi(\lambda)} + \frac{k}{\lambda}\right]\frac{\partial V}{\partial\lambda}\right\} \tag{2.6}$$

The solution of Equation $\Delta^* V$ dependent only on λ and vanishing at infinity is easily found

$$V = C\int_\lambda^\infty \frac{ds}{\sqrt{\omega(s)}}, \qquad \omega(s) = s^k\Psi(s) \tag{2.7}$$

Another more general solution is hence obtained by variation of the limits of integration. Beforehand we note that

$$(\nabla\Phi)^2 = \frac{d\Phi}{ds}, \qquad \Delta\Phi = -2\frac{\Psi'(s)}{\Psi(s)}$$

Now, let $g(t)$ be a twice differentiable function everywhere in the segment [0,1] with finite values of the one-sided derivatives at the boundary points. After slight manipulation we obtain the following formula for the operator Δ^* on the function $g(\Phi)$:

$$\Delta^* g(\Phi) = 4\sqrt{\omega(s)}\frac{d}{ds}\frac{g'(\Phi)}{\sqrt{\omega(s)}} \tag{2.8}$$

The form of this function suggests taking the following expression which satisfies the condition at infinity:

$$V = -\int_\lambda^\infty g(\Phi)\,du(s) = \int_\lambda^\infty g(\Phi)\frac{ds}{\sqrt{\omega(s)}} \qquad \left(u(s) = \int_s^\infty \frac{dt}{\sqrt{\omega(t)}}\right) \tag{2.9}$$

as a possible solution and, therefore, $u(\lambda)$ is a solution of the form of (2.7). Differentiating we obtain

$$\begin{aligned} \frac{\partial V}{\partial z} &= \int_\lambda^\infty \frac{\partial g(\Phi)}{\partial z}\frac{\partial s}{\sqrt{\omega(s)}} + g(0)\frac{\partial u(\lambda)}{\partial x}, \ldots \\ \frac{\partial^2 V}{\partial z^2} &= \int_\lambda^\infty \frac{\partial^2 g(\Phi)}{\partial z^2}\frac{ds}{\sqrt{\omega(s)}} - \frac{\partial g(\Phi)}{\partial z}\bigg|_{s=\lambda}\frac{1}{\sqrt{\omega(\lambda)}}\frac{\partial\lambda}{\partial x} + g(0)\frac{\partial^2 u(\lambda)}{\partial x^2} \end{aligned} \tag{2.10}$$

But
$$\frac{\partial g\,(\Phi)}{\partial x}\bigg|_{s=\lambda} = -\,g'\,(0)\,\frac{2x}{a^2+\lambda} = -\,4g'\,(0)\,\frac{\partial x}{\partial \lambda}\,,\;\ldots$$

Remarking that $\Delta^* u\,(\lambda) = 0$ and $\frac{\partial x}{\partial \lambda}\,\frac{\partial \lambda}{\partial x} + \ldots = 1$, upon adding we have

$$\Delta^* v = \int\limits_{\lambda}^{\infty} \Delta^* g\,(\Phi)\,\frac{ds}{\sqrt{\omega\,(s)}} + \frac{4g'\,(0)}{\sqrt{\omega\,(s)}} = 0 \tag{2.11}$$

identically because of (2.8). Thus the function (2.9) will actually be a solution. The last term in Expression (2.1) for $\Phi(s)$ vanishes on the boundary $\lambda = 0$ or $\mu = 0$. Hence,

$$V\,(x,\,y,\,0) = \int\limits_{\sigma}^{\infty} g\left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s}\right)\frac{ds}{\sqrt{\omega\,(s)}} \qquad \begin{cases}\sigma = 0 \text{ within the disk}\\ \sigma = \lambda \text{ outside the disk}\end{cases} \tag{2.12}$$

If $g(t)$ is a polynomial, then $V(x,\,y,\,0)$ is also a polynomial (with dependent coefficients) within the disk. Computing the derivatives, we obtain

$$-\,z^k\,\frac{\partial v}{\partial z} = 2\left(\frac{\mu\nu}{a^2b^2}\right)^{1/2\,(1+k)} \int\limits_0^1 h(\Phi_1)\,\frac{\xi^{1/2\,(1+k)}\,d\xi}{\sqrt{(\lambda + a^2\xi)\,(\lambda + b^2\xi)}} + \frac{z^k}{\sqrt{\omega\,(\lambda)}}\,\frac{\partial \lambda}{\partial z}$$
$$(h\,(t) = g'\,(t)) \tag{2.13}$$

Here Φ_1 is the result of the substitution $s = \lambda\xi^{-1}$ in Expression (2.1) for $\Phi(s)$, i.e.

$$\Phi_1 = \Phi_1\,(\xi) = (1 - \xi)\,\frac{(\lambda - \mu\xi)\,(\lambda - \nu\xi)}{(\lambda + a^2\xi)\,(\lambda + b^2\xi)}$$

It is clear that on the boundary outside the disk, $\mu = 0$, Expression (2.13) vanishes. Within the disk $\lambda = 0$, we have

$$f\,(x,\,y) = \frac{-1}{2\pi}\,z^k\,\frac{\partial V}{\partial z} = \frac{t^{1/2\,(1+k)}}{\pi ab}\int\limits_0^1 h\,(ut)\,(1-u)^{1/2\,(k-1)}\,du + \frac{g\,(0)}{\pi ab}\,t^{1/2\,(k-1)} \tag{2.14}$$

Here

$$t = \frac{\mu\nu}{a^2b^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \tag{2.15}$$

Hence, it has been proved that an elliptical distribution of the source density corresponds to the solution (2.10) of Equation (1.4). Moreover, it is seen that the density contains a component which is irregular on the boundary of the ellipse if $g(0) \neq 0$. There is just one component if $F(x,y) = c$. Then

$$f\,(x,\,y) = \frac{c}{\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2\,(k-1)}\left(\int\limits_0^{\infty}\frac{ds}{\sqrt{s^{1+k}\,(a^2+s)\,(b^2+s)}}\right)^{-1} \tag{2.16}$$

This result is presented in [3]. It has a somewhat different form in [4], to which it may be reduced if the substitution $s = t^{-1}$ is made in the integral of [2.16] and then contour integration is used.

Then we obtain as in [4]

$$f(x, y) = \frac{c}{\pi} \cos \frac{\pi k}{2} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2(k-1)} \left(\int_0^{\pi} \left(\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}\right)^{1/2(k-1)} d\varphi\right)^{-1} \tag{2.17}$$

Furthermore, using the notation $f(x, y) = q(t)$ in the case under consideration, let us substitute $ut = s$ in the integral. Then

$$q(t) = \frac{1}{\pi ab} \int_0^t h(s) (t-s)^{1/2(k-1)} ds + \frac{g(0)}{\pi ab} t^{1/2(k-1)} \tag{2.18}$$

Let us introduce a mass of sources Q . In our case

$$Q = \iint f(x, y)\, dS = \pi ab \int_0^1 q(t)\, dt = \frac{2}{1+k} \int_0^1 (1-s)^{1/2(1+k)} h(s)\, ds + \frac{2g(0)}{1+k} \tag{2.19}$$

For a known function $q(t)$ the dependence (2.18) is the Abel equation for $h(t)$ or, better, directly for $g(t)$. By the customary means (application of the Dirichlet formula), we obtain

$$g(t) = ab \cos \frac{\pi k}{2} \int_0^t q(s) (t-s)^{-1/2(1+k)} ds \tag{2.20}$$

This formula completely solves the problem of determining a function by means of a given source density $q(t)$ In the particular case when

$$q(t) = t^{1/2(k-1)} \sum_{n=0}^{m} c_n t^n \tag{2.21}$$

we have

$$g(t) = ab \cos \frac{\pi k}{2} \sum_{n=0}^{m} c_n \frac{\Gamma(1/2(1-k))\,\Gamma(1/2(2n+1+k))}{\Gamma(n+1)} t^n \tag{2.22}$$

As has been mentioned above, the dependences between the coefficients of the polynomial restrict the application of these results to the Dirichlet problem. This method permits its solution only when $F(x, y) = C - Ax - By$ (using differentiation with respect to x, y). It is also possible to solve a problem analogous to the problem of an elastic contact, i.e. to determine the size of an ellipse from the condition of finite pressure on its boundary. But this does not solve the problem of a nonplanar rigid stamp even in the case of a paraboloid.

3. The general, although preliminary approach to the formulated problem is obtained by separation of variables in (2.6). As a result, a single differential Equation

$$\frac{d^2E}{ds} + \frac{1}{2}\left\{\frac{1}{a^2+s} + \frac{1}{b^2+s} + \frac{1+k}{s}\right\}\frac{dE}{ds} = \frac{n(n+1+k)s - q}{4(a^2+s)(b^2+s)s} E \tag{3.1}$$

is obtained for the functions $\Lambda(\lambda)$, $M(\mu)$ and $N(\nu)$ whose product forms the solution.

Here $s = \lambda, \mu, \nu$ in the different ranges of variation of s . For convenience, one of the separation constants is written as $n(n+1+k)$. A comparison with the Heun equation in canonic form [7]

$$\frac{d^2w}{dz^2}+\left\{\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-a}\right\}\frac{dw}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-a)},\quad w=0 \tag{3.2}$$
$$(\gamma+\delta+\varepsilon=1+\alpha+\beta)$$

where q is an accessory parameter; shows that Equation (3.1) belongs to this type , having the exponents $(0,\ \frac{1}{2})$ in the strips $s=-a^2$, $s=-b^2$; $(0,\ \frac{1}{2}-\frac{1}{2}k)$ in the strip $s=0$ and $(-\frac{1}{2}n,\ \frac{1}{2}n+\frac{1}{2}+\frac{1}{2}k)$ at the infinitely distant point normal for this equation. Precisely this circumstance, which is due to the fact that the constant is $\gamma=0$ in the operator $\triangle^*$, eliminates the principal difficulty. By comparison with the Lamé equation, Equation (3.1) has no uniformizing substitution since the integral

$$\int^{u}(a^2+s)^{1/2}(b^2+s)^{1/2}s^{1/2(1+k)}\,ds$$

does not admit of a unique inversion. However, in the real domain this does not imply difficulties in principle and the transcendental coordinates ξ,η,ζ may be introduced exactly as is done for the Lamé equation. These parameters are not used herein. Here it is expedient to transform from the symmetric form of the λ,μ,ν coordinates, convenient for the development of the analytic theory of the Lamé equation, to the nonsymmetric ρ,μ,ν coordinates by making the substitution $s+a^2=\sigma^2$. Equation (2.1) then becomes

$$\Phi(\sigma^2)=1-\frac{x^2}{\sigma^2}-\frac{y^2}{\sigma^2-c^2}-\frac{z^2}{\sigma^2-a^2}=0 \qquad (c=\sqrt{a^2-c^2}) \tag{3.3}$$

(in [8] the quantities c, a are denoted by h, k). The transformation formulas to Cartesian Coordinates and Lamé coefficients are

$$x=\frac{\rho\mu\nu}{ac},\qquad y=\frac{\sqrt{\rho^2-c^2}\sqrt{\mu^2-c^2}\sqrt{c^2-\nu^2}}{c\sqrt{a^2-c^2}};\qquad z=\frac{\sqrt{\rho^2-a^2}\sqrt{a^2-\mu^2}\sqrt{a^2-\nu^2}}{a\sqrt{a^2-c^2}};$$
$$H_\rho=\left(\frac{(\rho^2-\mu^2)(\rho^2-\nu^2)}{(\rho^2-a^2)(\rho^2-c^2)}\right)^{1/2},\quad H_\mu=\left(\frac{(\rho^2-\mu^2)(\mu^2-\nu^2)}{(a^2-\mu^2)(\mu^2-c^2)}\right)^{1/2},\quad H_\nu=\left(\frac{(\rho^2-\nu^2)(\mu^2-\nu^2)}{(a^2-\nu^2)(c^2-\nu^2)}\right)^{1/2} \tag{3.4}$$

On the disk $\rho=a$ we have

$$x=\frac{\mu\nu}{c},\qquad y=\frac{\sqrt{(\mu^2-c)(c^2-\nu^2)}}{c},\qquad H_\mu=\left(\frac{\mu^2-\nu^2}{\mu^2-c^2}\right)^{1/2}$$
$$H_\nu=\left(\frac{\mu^2-\nu^2}{c^2-\nu^2}\right)^{1/2},\qquad dS=\frac{(\mu^2-\nu^2)\,d\mu\,d\nu}{\sqrt{(\mu^2-c^2)(c^2-\nu^2)}} \tag{3.5}$$

Equation (3.1) is transformed to

$$(\sigma^2-a^2)(\sigma^2-c^2)\frac{d^2E}{d\sigma^2}+\sigma\left[(\sigma^2-a^2)+(1+k)(\sigma^2-c^2)\right]\frac{dE}{d\sigma}+$$
$$+\left[q-n(n+1+k)\sigma^2\right]E=0 \tag{3.6}$$

The indices relative to the infinitely distant point for this equation are $(-n,n+1+k)$ and Equation (2.6) admits of a polynomial solution for the characteristic values of the parameter q

$$E = K(\sigma) + \sum_{s=0}^{r} a_s \sigma^{n-2s} \qquad \left(r = \left[\frac{1}{2} n\right]\right) \tag{3.7}$$

and a quasi-polynomial

$$E = L(\sigma) = \sqrt{\sigma^2 - c^2} \sum_{s=0}^{r} b_s \sigma^{n-1-2s} \qquad \left(r = \left[\frac{n-1}{2}\right]\right) \tag{3.8}$$

Substitution of (3.7) into (3.6) leads to the following recursion system for the coefficients

$$\begin{aligned} 2(s+1)(2n+2s-1+k)\, a_{s+1} = \{q - (n-2s)(n-2s-1)\, d^2 - \\ - (n-2s) f^2\}\, a_s + (n-2s+2)(n-2s+1)\, e^4 a_{s-1} \end{aligned} \tag{3.9}$$

$$d^2 = a^2 + c^2, \qquad e^4 = a^2c^2, \qquad f^2 = d^2 + kc^2$$

Putting $s = r+1$, $a_{r+1} = 0$, we obtain $a_{r+2} = 0$ and then all the $a_s = 0$ for $s > r$. Equating the determinant of the homogeneous system for the remaining s to zero, we obtain an equation of $r+1$ degree in q.

Hence, $\frac{1}{2}n + 1$ functions $K(\sigma)$ are obtained if n is even and $\frac{1}{2}(n+1)$ of the same functions if n is odd (absence of multiple roots is assumed, as actually occurs). In exactly the same manner for the coefficients b_s, the substitution of (3.8) into (3.6) leads to the system

$$\begin{aligned} 2(s+1)(2n-2s-1+k)\, b_{s+1} = \{q_1 - (n-2s-1)(n-2s-2) \times \\ \times d^2 - (n-2s-1) g^2\}\, b_s - (n-2s+1)(n-2s)\, e^4 b_{s-1} \end{aligned} \tag{3.10}$$

$$g^2 = 3a^2 + (1-k)\, c^2, \qquad q_1 = q - a^2$$

Repeating the previous reasoning, we find that the characteristic equation for q is of $\frac{1}{2}n$ degree if n is even and of $\frac{1}{2}(n+1)$ degree if n is odd. Hence, the total number of functions of both kinds K, L of degree n is $n+1$. The products $K(\mu)K(\nu)$ and $L(\mu)L(\nu)$ are polynomials in x, y which have the same degree. The total number of functions of degree from 0 to n, inclusively, is $\frac{1}{2}(n+1)(n+2)$, i.e. agrees with the number of independent elements of the basis of nth degree polynomials in x, y. Under the condition of linear independence of the products $E(\mu)E(\nu)$, they can be represented by a linear combination of arbitrary nth degree polynomials in x, y. But exactly as is done in the theory of ellipsoidal functions [8], it is easy to prove the orthogonality of these products, hence, their linear depencence indeed follows. Let E_n^s denote a function of degree n which belongs to the eigenvalue q_n^s, a root of the characteristic nth degree equation. The Wronskian

$$H = E_{n'}^{s'} \frac{dE_n^s}{d\sigma} - E_n^s \frac{dE_{n'}^{s'}}{d\sigma} \tag{3.11}$$

of functions of the same kind will be polynomial in both cases. It will satisfy the first order differential Equation

$$\frac{d}{d\sigma}\{\Delta(\sigma) H\} = \varepsilon \frac{[n(n+1+k) - n'(n'+1+k)]\,\sigma^2 - \left(q_n^s - q_{n'}^{s'}\right)}{\sqrt{|\sigma^2 - c^2|\,|\sigma^2 - a^2|^{1-k}}}\, \varepsilon_n^s\, \varepsilon_{n'}^{s'} \tag{3.12}$$

where

$$\Delta(\sigma) = |\sigma^2 - c^2|^{1/2} |\sigma^2 - a^2|^{1/2(1+k)}, \qquad \varepsilon = \operatorname{sign}(\sigma^2 - a^2)(\sigma^2 - c^2)$$

Integrating this equality over the intervals $c \leqslant \sigma \leqslant a$ and $-c \leqslant \sigma \leqslant c$, at whose endpoints $\Delta(\sigma) = 0$, we obtain the relations

$$[n(n+1+k) - n'(n'+1+k)] \int_c^a \frac{\mu^2 E_n^s(\mu) E_{n'}^{s'}(\mu)\, d\mu}{\sqrt{(\mu^2 - c^2)(a^2 - \mu^2)^{1-k}}} =$$

$$= \left(q_n^s - q_{n'}^{s'}\right) \int_c^a \frac{E_n^s(\mu) E_{n'}^{s'}(\mu)\, d\mu}{\sqrt{(\mu^2 - c^2)(a^2 - \mu^2)^{1-k}}}$$

$$[n(n+1+k) - n'(n'+1+k)] \int_{-c}^c \frac{\nu^2 E_n^s(\nu) E_{n'}^{s'}(\nu)\, d\nu}{\sqrt{(c^2 - \nu^2)(a^2 - \nu^2)^{1-k}}} = \tag{3.13}$$

$$= \left(q_n^s - q_{n'}^{s'}\right) \int_{-c}^c \frac{E_n^s(\nu) E_{n'}^{s'}(\nu)\, d\nu}{\sqrt{(c^2 - \nu^2)(a^2 - \nu^2)^{1-k}}}$$

Cross-multiplying these equalities and substracting, we find that the integral

$$J_{n,n'}^{s,s'} = \int_c^a \int_{-c}^c \frac{(\mu^2 - \nu^2) E_n^s(\mu) E_n^s(\nu) E_{n'}^{s'}(\mu) E_{n'}^{s'}(\nu)\, d\mu\, d\nu}{\sqrt{(\mu^2 - c^2)(c^2 - \nu^2)(a^2 - \mu^2)^{1-k}(a^2 - \nu^2)^{1-k}}} \tag{3.14}$$

vanishes if $n = n'$ and $q_n^s = q_{n'}^{s'}$ simultaneously. But because the single integrals (3.13) vanish for $n = n'$ and $q_n^s \neq q_{n'}^{s'}$ or $n \neq n'$ and $q_n^s = q_{n'}^{s'}$, the double integral (3.14) is also zero in these cases.

Thus, the integral (3.14) is not zero only when $n = n'$ and $q_n^s = q_{n'}^{s'}$ simultaneously. This integral is positive because $\mu^2 > \nu^2$

Therefore, there is a real normalization of the products $E_n^s(\mu) E_n^s(\nu)$ such that these products from an orthonormal system. Transforming to Cartesian coordinates by means of (3.5) and using the notation

$$P_n^s(x, y) = E_n^s(\mu) E_n^s(\nu),$$

we obtain

$$\iint \frac{P_n^s(x, y) P_{n'}^{s'}(x, y)\, dx\, dy}{(1 - x^2/a^2 - y^2/b^2)^{1/2(1-k)}} = 0 \tag{3.15}$$

The integral is taken over the area of the ellipse

$$1 - x^2/a^2 - y^2/b^2 > 0.$$

Thus the polynomials $P_n^s(x,y)$ form an orthogonal system with weight $(1 - x^2/a^2 - y^2/b^2)^{1/2(k-1)}$ in the area of the ellipse. They thereby differ from the V.A. Steklov polynomials [9] which form an orthogonal system with weight $(1 - x^2/a^2 - y^2/b^2)^{\alpha}$ on the area of the ellipse, whereby $\alpha > 0$.

The remaining propositions on the theory of the functions $E_n^s(\mu)$ are not developed here. This theory, as is seen, duplicates the theory of ellipsoidal functions with insignificant changes, hence, the proof that all the eigenvalues are real and different and that the roots of the functions $E_n^s(\mu)$ are enclosed in the interval $(-a, a)$, is also omitted.

Let us note only that the evaluation of the integral $J_{n,n'}^{s,s'}$ which is needed for the normalization, reduces to just rational operations, exactly as in the theory of ellipsoidal functions [8] because the integrals

$$K_r = \int \sigma^{2r} \, | \sigma^2 - c^2 |^{-1/2} \, | \sigma^2 - a^2 |^{1/2(k-1)} \, d\sigma \qquad (r \text{ a natural number})$$

taken within the limits (c, a) or $(0, c)$, are expressed by linear combinations of the integrals K_0, K_1 (which are identical for both ranges of integration) via the reduction formulas. Hence, the integral (3.14) equals the product of rational functions of a, c and the integral

$$\int_c^a \int_0^c \frac{(\mu^2 - \nu^2)\, d\mu d\nu}{\sqrt{(\mu^2 - c^2)(c^2 - \nu^2)(a^2 - \mu^2)^{1-k}(a^2 - \nu^2)^{1-k}}}$$

Transforming to Cartesian coordinates by means of (3.5), we remark that the integral is evaluated by elementary means and equals $\pi (1 + k)^{-1} (ab)^k$.

4. Let us now deduce the integral equation for the products

$$P_n^s(x, y) = E_n^s(\mu) E_n^s(\nu)$$

by using Formulas (1.5) and (1.6).

Let us introduce a function of the second kind $F_n^s(\rho)$. This is the second solution of Equation (3.6) in the interval $\sigma \geqslant \rho$ (where the first solution is $E_n^s(\rho)$) subject to condition $F_n^s(\rho) \to 0$ as $\rho \to \infty$. Evidently the solution is expressed by Formula (constant factor is omitted)

$$F_n^s(\rho) = E_n^s(\rho) \int_\rho^\infty \frac{d\sigma}{[E_n^s(\rho)]^2 \Delta(\sigma)} \tag{4.1}$$

Let $V(x, y, 0) = P_n^s(x, y)$ on the disk. The function $V(x, y, z)$, a solution of Equation (1.4), is determined outside the disk by Formula

$$V(x, y, z) = \frac{F_n^s(\rho)}{F_n^s(a)} E_n^s(\mu) E_n^s(\nu) \tag{4.2}$$

Evaluating the derivative $\partial V/\partial z$ by means of (3.4), we find

$$\frac{\partial V}{\partial z} = \frac{\sqrt{(\rho^2 - a^2)(a^2 - \mu^2)(a^2 - \nu^2)}}{a\sqrt{a^2 - c^2}} \left\{ \frac{(\rho^2 - c^2)\rho}{(\rho^2 - \mu^2)(\rho^2 - \nu^2)} \frac{\partial V}{\partial \rho} - \right.$$
$$\left. - \frac{(\mu^2 - c^2)\mu}{(\rho^2 - \mu^2)(\mu^2 - \nu^2)} \frac{\partial V}{\partial \mu} - \frac{(c^2 - \nu^2)\nu}{(\rho^2 - \nu^2)(\mu^2 - \nu^2)} \frac{\partial V}{\partial \nu} \right\} \tag{4.3}$$

Hence

$$\lim_{\rho \to a} z^k \frac{\partial v}{\partial z} = \lim z^k \frac{\rho(\rho^2 - c^2)\sqrt{(\rho^2 - a^2)(a^2 - \mu^2)(a^2 - \nu^2)}}{a\sqrt{a^2 - c^2}(\rho^2 - \mu^2)(\rho^2 - \nu^2)} \frac{\partial v}{\partial \rho} =$$
$$= a^{-k} \left[\frac{(a^2 - \mu^2)(a^2 - \nu^2)}{(a^2 - c^2)} \right]^{1/2(k-1)} \lim_{\rho \to a} (\rho^2 - a^2)^{1/2(1+k)} \frac{\partial v}{\partial \rho} \tag{4.4}$$

But since

$$\frac{\partial v}{\partial \rho} = \frac{E_n^s(\mu) E_n^s(\nu)}{F_n^s(a)} \frac{dF_n^s(\rho)}{d\rho}$$

$$\frac{dF_n^s(\rho)}{d\rho} = \frac{dE_n^s(\rho)}{d\rho} \int\limits_\rho^\infty \frac{d\sigma}{[E_n^s(\sigma)]^2 \Delta(\sigma)} - \frac{1}{[E_n^s(\rho)]^2 \Delta(\rho)} \tag{4.5}$$

then

$$\lim_{\rho \to a} z^k \frac{\partial V}{\partial z} = - \frac{E_n^s(\mu) E_n^s(\nu)}{E_n^s(a) F_n^s(a)} \frac{[(a^2 - \mu^2)(a^2 - \nu^2)]^{1/2(k-1)}}{a^k (a^2 - c^2)^{1/2 k}} \tag{4.6}$$

Evaluating the source density by means of (1.6), we obtain in Cartesian coordinates

$$f(x, y) = \frac{(1 - x^2/a^2 - y^2/b^2)^{1/2(k-1)}}{2\pi ab E_n^s(a) F_n^s(a)} P_n^s(x, y) \tag{4.7}$$

Expressing the value $v(x, y, 0)$ in terms of $f(x, y)$, we obtain

$$P_n^s(x, y) = \frac{1}{2\pi ab E_n^s(a) F_n^s(a)} \iint \frac{(1 - \xi^2/a^2 - \eta^2/b^2)^{1/2(k-1)} P_n^s(\xi, \eta)\, d\xi\, d\eta}{[(x-\xi)^2 + (y-\eta)^2]^{1/2(1+k)}}$$

This is indeed the desired integral equation. On the basis of the general theory, it is now possible to write at once the series expansion of r^{-1-k} in products of the eigenfunctions. But there is no need for this expansion.

More important is the correspondence between the value of $V(x, y, 0) = P_n^s(x, y)$ and $f(x,y)$ by means of (4.7). If $V(x,y,0) = F(x,y)$ is an arbitrary nth degree polynomial, then by expanding it in the products $P_m^s(x, y) = E_m^s(\mu) E_m^s(\nu)$ and comparing the density $f_n^s(x,y)$ to each of the members of the expansion by means of (4.7), we will find that $f(x, y) = f_0(x, y) + f_1(x, y) + \ldots + f_n(x, y)$ equals the product of a polynomial of the same degree by the function $(1 - x^2/a^2 - y^2/b^2)^{1/2(k-1)}$. The theorem inverse to the theorem given in [4] is thereby proved, i.e. it has been proved that if a polynomial is on the left-hand side of Equation (0.1), then the solution of the equation is a polynomial of the same degree, but with other coefficients, multiplied by $(1 - x^2/a^2 - y^2/b^2)^{1/2(k-1)}$.

This theorem is an analog of a theorem of Galin [10] on the pressure of am elliptical planform stamp on an elastic half-space. The theorem of [4] and this theorem, together, yield a basic for the application of the method of undetermined coefficients to the solution of Equation (0.1). If evaluations of the function $w(x,y)$ outside the limits of the ellipse are not required, then calculation according to [4] yield the solution by simpler means than the evaluation of the expansion of $V(x,y,0)$ in $P_n^s(x,y)$.

When the evaluation of $w(x,y)$ outside the boundaries of the ellipse is necessary, the formulas of Sections 3 and 4 herein yield a finite algorithm of the solution (under the assumption that $w(x,y)$ is a polynomial).

5. Let us derive formulas connecting the parameters of rigid displacement of a stamp with loading Q and moments M_x, M_y acting on the stamp

$$Q = \iint f(x, y)\, dx\, dy, \qquad M_x = \iint y f(x, y)\, dx\, dy, \qquad M_y = \iint x f(x, y)\, dx\, dy \tag{5.1}$$

Let us assume that the settling $w(x,y)$ is represented by Expression

$$w(x, y) = \delta + \alpha x + \beta y + W(x, y) \qquad (W(x, y) = O(r^2)) \tag{5.2}$$

The values of the potential $V(x,y,0)$ differ from $w(x,y)$ by the constant factor K (stiffness of the sole) which has the [force (length)$^{-2-k}$] dimension

$$V(x, y, 0) = Kw(x, y) \tag{5.3}$$

In general, K is an empiric coefficient. For a homogeneous elastic half-space $K = (\pi E)/(1 - \nu^2)$.

Expanding $w(x,y)$ and $f(x,y)$ in series of eigenfunctions $P_n^s(x,y)$, we have in conformity with (4.4)

$$w(x, y) = \sum_{n=0}^{\infty} \sum_{s=0}^{n+1} C_n^s P_n^s(x, y) \tag{5.4}$$

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{s=0}^{n+1} \frac{Kc_n^s P_n^s(x, y)}{2\pi ab E_n^s(a) E_n^s(a)} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2(k-1)}$$

As a consequence of the orthogonality of the family $P_n^s(x,y)$ we obtain

$$\iint f(x, y) P_n^s(x, y)\, dx\, dy =$$

$$= \frac{K}{2\pi ab E_n^s(a) F_n^s(a)} \iint w(x, y) P_n^s(x, y) \frac{dx\, dy}{(1 - x^2/a^2 - y^2/b^2)^{1/2(1-k)}} \tag{5.5}$$

(integrals taken over the area of the ellipse)

Let us apply this formula to lower degree polynomials

$$P_0^\circ(x, y) = 1, \qquad P_1^\circ(x, y) = x, \qquad P_1^1(x, y) = y$$

We obtain

$$Q = \frac{K}{2\pi ab E_0^\circ(a) F_0^\circ(a)} \iint \frac{w(x, y)\, dx\, dy}{(1 - x^2/a^2 - y^2/b^2)^{1/2(1-k)}} \tag{5.6}$$

$$M_x = \frac{K}{2\pi ab E_1^1(a) F_1^1(a)} \iint \frac{yw(x, y)\, dx\, dy}{(1 - x^2/a^2 - y^2/b^2)^{1/2(1-k)}} \tag{5.7}$$

$$M_y = \frac{K}{2\pi ab E_1^\circ(a) F_1^\circ(a)} \iint \frac{xw(x, y)\, dx\, dy}{(1 - x^2/a^2 - y^2/b^2)^{1/2(1-k)}} \tag{5.8}$$

Keeping in mind the corresponding results of Galin in the three-dimensional problem of the theory of elasticity [10], let us use his notation for the coefficients in front of the integrals in the above formulas

$$A = K\, [2\pi ab E_1^\circ(a)\, F_1^\circ(a)]^{-1}, \qquad B = K\, [2\pi ab E_1^1(a)\, F_1^1(a)]^{-1}$$
$$C = K\, [2\pi ab E_0^\circ(a)\, F_0^\circ(a)]^{-1} \tag{5.9}$$

Moreover, let us use the notation

$$J_1 = \iint x^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2(k-1)} dx\,dy = \frac{2\pi a^3 b}{(1+k)(3+k)}$$

$$J_2 = \iint y^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2(k-1)} dx\,dy = \frac{2\pi a b^3}{(1+k)(3+k)} \tag{5.10}$$

$$J_0 = \iint \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2(k-1)} dx\,dy = \frac{2\pi ab}{1+k}$$

Inserting Expressions (5.4) into Formulas (5.6) to (5.8) and solving the obtained equations for δ, α, β we find

$$\delta = \frac{1}{CJ_0}\left\{Q - C\iint \frac{W(x,y)\,dx\,dy}{(1-x^2/a^2-y^2/b^2)^{1/2(1-k)}}\right\} \tag{5.11}$$

$$\alpha = \frac{1}{AJ_1}\left\{M_y - A\iint \frac{xW(x,y)\,dx\,dy}{(1-x^2/a^2-y^2/b^2)^{1/2(1-k)}}\right\} \tag{5.12}$$

$$\beta = \frac{1}{BJ_2}\left\{M_x - B\iint \frac{yW(x,y)\,dx\,dy}{(1-x^2/a^2-y^2/b^2)^{1/2(1-k)}}\right\} \tag{5.13}$$

These formulas which express the parameters of rigid displacement as a function of the force Q and the moments M_x, M_y, have the same form as the corresponding formulas of Galin, but the values of the constants are different here and the degree of the weight function in the integrand is here $\frac{1}{2}(k-1)$ instead of $-\frac{1}{2}$. Let us express the coefficients A, B, C explicitly in terms of integrals.

Returning to Formula (4.1), we note that the product $E_n^s(a)\,F_n^s(a)$ is independent of the constant factor in the expression for $E_n^s(\rho)$.

In this connection, let us assume for simplicity

$$E_0^{\circ}(\rho) = 1, \qquad E_1^{\circ}(\rho) = \rho, \qquad E_1^{1}(\rho) = \sqrt{\rho^2 - c^2} \tag{5.14}$$

Substitution of these functions into (4.1) yields

$$E_0^{\circ}(a)\,F_0^{\circ}(a) = \int_a^{\infty} \frac{d\sigma}{\sqrt{(\sigma^2-c^2)(\sigma^2-a^2)^{1+k}}}$$

$$E_1^{\circ}(a)\,F_1^{\circ}(a) = \int_a^{\infty} \frac{a^2\,d\sigma}{\sigma^2\sqrt{(\sigma^2-c^2)(\sigma^2-a^2)^{1+k}}} \tag{5.15}$$

$$E_1^{1}(a)\,F_1^{1}(a) = \int_a^{\infty} \frac{(a^2-c^2)\,d\sigma}{(\sigma^2-c^2)\sqrt{(\sigma^2-c^2)(\sigma^2-a^2)^{1+k}}}$$

Making the change of variable $\sigma = at$ and introducing the results into (5.9), we obtain

$$A = \frac{Ka^k}{2\pi b\psi_1}, \qquad B = \frac{Ka^k}{2\pi b\psi_2(1-c^2)}, \qquad C = \frac{Ka^k}{2\pi b\psi_0}$$

where the ψ_0, ψ_1 and ψ_2 are nondimensional coefficients expressed via the integrals

$$\psi_0 = \int_1^\infty \frac{dt}{\sqrt{(t^2-e^2)(t^2-1)^{1+k}}}, \qquad \psi_1 = \int_1^\infty \frac{dt}{t^2\sqrt{(t^2-e^2)(t^2-1)^{1+k}}} \tag{5.16}$$

$$\psi_2 = \int_1^\infty \frac{dt}{(t^2-e^2)\sqrt{(t^2-e^2)(t^2-1)^{1+k}}}$$

Here $e = c/a$ is the eccentricity of the ellipse. For $k = 0$ and $K = \pi E/(1-\nu^2)$ Formulas (5.15), (5.16) and (5.10) transform into formulas (9.35) of the monograph [10]. If the stamp is a plane, i.e. $W = 0$, then Formulas (5.11) to (5.13) are simplified. Taking into account (5.10) and (5.15) we obtain in this case

$$\delta = \psi_0 \frac{(1+k)\,Q}{Ka^{1+k}}, \qquad \alpha = \psi_1 \frac{(1+k)(3+k)}{Ka^{3+k}} M_y, \qquad \beta = \psi_2 \frac{(1+k)(3+k)}{Ka^{3+k}} M^x \tag{5.17}$$

In order to have the possibility of applying these results to estimating the displacement δ of a flat stamp of arbitrary planform, it is necessary to extend the maximum principle to the function V, the solution of Equation $\Delta^* V = 0$ which it is expedient to consider in the whole space with the slit along the plane domain $z = 0$ occupied by sources. In this space the function V defined by (1.5) evidently has no local extrema for $z \neq 0$. But under the condition $f(x,y) > 0$, the function V has no local extrema also for $z = 0$. This can be seen by evaluating the second derivatives at the point $(x_0, y_0, 0)$. The characteristic equation of the matrix of the second derivatives has two negative and one positive root under this condition. Therefore, local extrema are missing at these points also. From this same analysis it follows that the surface $z = V(x,y,0)$ consists of hyperbolic points outside the cut.

As applied to a flat stamp, this means that if the stamp is moved without having any deflection ($\alpha = \beta = 0$), then $w(x,y) < \delta$ everywhere beyond the stamp. Let S denote the domain under the stamp and E the interior of an ellipse such that $E \supset S$. Then

$$w = \delta \quad \text{on } S; \qquad w = g(x,y) < \delta \quad \text{on } E - S \tag{5.18}$$

Now, applying Formula (5.6) to the domain E, we find

$$Q < CJ_0\delta = \frac{Ka^{1+k}}{(1+k)\psi_0}\,\delta \tag{5.19}$$

In order to obtain an exact upper bound for Q, evidently that one of all the ellipses $E \supset S$ should be chosen on which the ratio a^{1+k}/ψ_0 achieves the exact lower bound.

6. Let us now consider the limit case when the elliptical domain becomes circular and the ρ, μ, ν coordinate system degenerates into the η, θ, φ coordinate system of an oblate spheroid

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z + ir = a\sinh(\eta + i\theta) \tag{6.1}$$

In this passage to the limit on the vanishing section $\sigma = \nu$, $0 \leqslant \nu \leqslant c$, $c \to 0$ Equation (3.6) yields

$$\frac{d^2E}{d\varphi^2} + m^2E = 0, \qquad q = m^2a^2 \tag{6.2}$$

and the functions $E(\nu)$ transform in [8] into $\cos m\varphi$, $\sin m\varphi$. On the remaining sections we have

$$(\zeta^2-1)\,\zeta^2\frac{d^2E}{d\zeta^2} + \zeta\,[(2+k)\,\zeta^2 - 1]\,\frac{dE}{d\zeta} + [m^2 - n(n+1+k)\,\zeta^2]\,E = 0 \tag{6.3}$$

$$\zeta = \cosh\eta \quad \text{for } \zeta > 1, \qquad \zeta = \sin\theta \quad \text{for } 0 < \zeta < 1 \qquad (\zeta = \sigma/a)$$

Equations (6.2) and (6.3) may be obtained directly also by separation of Equation $\triangle^* V = 0$ in the η, θ, φ coordinates, which yields

$$\frac{d^2H}{d\eta^2} + (\tanh\eta + k\coth\eta)\frac{dH}{d\eta} + \left(-h + \frac{m^2}{\cosh^2\eta}\right)H = 0 \qquad (h = n(n+1+k)) \quad (6.4)$$

$$\frac{d^2\Theta}{d\theta^2} + (\cot\theta - k\tan\theta)\frac{d\Theta}{d\theta} + \left(h - \frac{m^2}{\sin^2\theta}\right)\Theta = 0$$

The substitutions $\zeta = \cosh\eta$, $\zeta = \sin\theta$ lead to (6.3). But for an isolated analysis, the relation to the general theory presented above is touched upon. The passage to the limit makes clear why the results for a circular domain are expressed in comparatively simple hypergeometric functions while the other case of degeneration (infinite strip, elliptic cylindrical coordinate system) requires involvement with Mathieu functions. In the first confluent case a decrease in the number of singular points in the Heun equation (3.1) occurs and the infinitely distant point remains regular; in the second case we obtain an irregular point at infinity even when $k = 0$ and $\Delta V = 0$ (in the problem of the theory of elasticity for half-space).

The basic results for the circular domain are known. But they have been obtained without relation to the theory of equations of Fuchs type, by means of direct, sometimes very complicated calculations [12], by methods which are very diverse in their conceptual bases. Let us show here how these results may be obtained on the basis of the general theory developed.

Let us note at once that $c = 0$ implies directly the transformation of the systems of three-term equations (3.10) and (3.11) into a system of two-term equations for the coefficients of the hypergeometric series. Equation (6.3) is reduced to

$$\frac{d^2E}{du^2} + \left(\frac{1}{u} + \frac{1/2 + 1/2k}{u-1}\right)\frac{dE}{du} + \frac{m^2 - n(n+1+k)u}{4u^2(u-1)}E = 0 \qquad (6.5)$$

by the substitution $\zeta = u^2$.

Evaluating the exponents, we find that the scheme of solving this equation will be [13]

$$E = P\begin{Bmatrix} 0, & 1, & \infty, & \\ 1/2 m, & 0, & -1/2 n, & u \\ -1/2 m, & 1/2 - 1/2 k, & 1/2(n+1-k), & \end{Bmatrix} \qquad (6.6)$$

Performing the reduction to the standard scheme of the hypergeometric equation we obtain

$$E = u^{1/2 m} P\begin{Bmatrix} 0, & 1, & \infty, & \\ 0, & 0, & a, & u \\ 1-c, & c-a-b, & b, & \end{Bmatrix} \quad \begin{pmatrix} a = 1/2(m-n) \\ b = 1/2(m+n+1+k) \\ c = m+1 \end{pmatrix} \qquad (6.7)$$

Limiting ourselves to the case of analytic $V(x,y,0)$, we note that n and m are non-negative integers, $n \geqslant m$ and the difference $n - m$ is an even number. Equation (6.5) then has the polynomial solution

$$E_n^m = u^{1/2 m} F(1/2(m-n), 1/2(m+n+1+k), m+1; u) \qquad (6.8)$$

To the accuracy of a constant factor the function F agrees with the Jacobi polynomial

$$P_n^{(\alpha,\beta)}(x) \text{ where } \alpha = m, \qquad \beta = 1/2(k-1), \qquad x = 1 - 2u^2$$

as follows from the formulas representing Jacobi polynomials by means of hypergeometric functions [7] and [14]. Thus

$$E_n^m(\sin\theta) = (\sin\theta)^m P_{1/2(n-m)}^{(m, 1/2(k-1))}(\cos 2\theta) \qquad (6.9)$$

can be taken as the normal form of the solution.

Since Equation (1.4) transforms into the equation $\Delta V = 0$ for $k = 0$, the functions (6.9) transform into the associated Legendre functions $P_n^m(\cos\theta)$ for $k = 0$. This is easily verified by applying Theorem (4.1) from [14] and the formula for differentiating Jacobi polynomials.

From the weight formula of the Jacobi polynomials $w(x) = (1-x)^\alpha (1+x)^\beta$ we find the weight of the functions E_n^m by simple transformation

$$w(\zeta) = \zeta(1-\zeta^2)^{1/2(k-1)} \qquad (0 \leqslant \zeta \leqslant 1) \tag{6.10}$$
$$w(\theta) = \sin\theta \cos^k\theta \qquad (0 \leqslant \theta \leqslant {}^1/_2\pi)$$

These formulas are easily verified directly by means of Equations (6.3) and (6.4). The above is sufficient for the solution of the Dirichlet problem. By taking the normal solution in the form

$$V_{n,m} = \frac{F_n{}^m(\operatorname{ch}\eta)}{F_n{}^m(1)} E_n{}^m(\sin\theta)\sin\cos m\varphi \tag{6.11}$$

a computation may be made which is analogous to that made in Section 4, and the eigenvalues may be found for the functions $E_n{}^m(r/a)$ of the integral equation

$$F_m(r) = \int_0^a sf(s)\,ds \int_0^{2\pi} \frac{\cos m\omega d\omega}{(r^2+s^2-2rs\cos\omega)^{1/2(1+k)}} \tag{6.12}$$

which is obtained from (0.1) by expanding the functions in Fourier series in $\cos m\varphi$ and $\sin m\varphi$. Let us note that the equation for $E_n{}^m(r/a)$ differs from (6.12) by weight factor $(1-s^2/a^2)^{1/2(k-1)}$.

Hence, it has been shown that the problem for the circular domain is included in the more general theory developed in Sections 1 to 4. The eigenvalues for the functions E_n^m are found in [12].

7. Let us consider a generalization of equation (6.12) which is associated with the fact that its kernel is represented by different hypergeometric functions on the sections $0 < s < r$ and $r < s < a$ of the interval of integration, namely [7]

$$\int_0^{2\pi} \frac{\cos n\omega\, d\omega}{(x^2+y^2-2xy\cos\omega)^{1/2(1+r)}} = \frac{2\pi\Gamma(n+{}^1/_2(1+r))}{\Gamma({}^1/_2(1+r))\,\Gamma(n+1)} \times$$
$$\times x^n y^{-n-r-1} F\left(n+\frac{1+r}{2},\ \frac{1+r}{2};\ n+1,\ \frac{x^2}{y^2}\right) \tag{7.1}$$

if $x^2 < y^2$. For $x^2 > y^2$ the letters x and y exchange places. (It is here convenient to change the special notations of the preceding sections to general notations). Representation of (7.1) by an Euler integral was used in [4] to obtain the solution in closed form. The same function may be represented as

$$\frac{2^{1-r}\Gamma({}^1/_2(1-r))}{\Gamma({}^1/_2(1+r))}\pi \int_0^\infty J_n(xt)\,J_n(yt)\,t^r\,dt \tag{7.2}$$

Such a representation has been used in [11] and [12] for the same purpose. The solution is then obtained either by applying the Barnes integrals [11] or by the reduction to a Wiener-Hopf equation with the subsequent use of the method of M.G. Krein [12]. A natural generalization of the form of the kernel (7.2) is the discontinuous Weber-Schafheitlin integral [7]

$$W_{p,q}^{(r)}(x,y) = \int_0^\infty J_p(xt)\,J_q(yt)\,t^r\,dt = 2^r x^p y^{-1-r-p} \frac{\Gamma({}^1/_2(1+r+p+q))}{\Gamma({}^1/_2(1-r+q-p))\,\Gamma(1+p)} \times$$
$$\times F\left(\frac{1+r+p+q}{2},\ \frac{1+r+p-q}{2};\ 1+p;\ \frac{x^2}{y^2}\right) \tag{7.3}$$
$$x^2 < y^2, \qquad \operatorname{Re}(p+q+r+1) > 0, \qquad \operatorname{Re} r < 1$$

The second formula is obtained by permutation of the letters $(x,p) \rightleftarrows (y,q)$. An equation of the form

$$f(x) = \int_0^a W_{p,q}^{(r)}(x, y)\, \varphi(y)\, dy \tag{7.4}$$

was considered in [12] from the viewpoint of the Wiener-Hopf and the Krein methods. But it is known that any equation of Wiener-Hopf type may be reduced to two duplicate Volterra equations. This "Copson method" seems very simple and powerful in a number of cases and has been successfully used by various authors [15], including the present author in [4] and other works. In this case this method leads to the inversion formula

$$\varphi(x) = \frac{-2^{1-r}x^q}{\Gamma(^1/_2(1+r+p+q))\,\Gamma(^1/_2(1+r+q-p))} \times$$
$$\times \frac{d}{dx}\int_x^a \left\{ \frac{t^{1-r-p-q}}{(t^2 - x^2)^{^1/_2(1-r+q-p)}} \frac{d}{dt}\int_0^t \frac{u^{1+p} f(u)\, du}{(t^2 - u^2)^{^1/_2(1-r+p-q)}} \right\} dt \tag{7.5}$$

which holds under the following sufficient conditions: (1) $\operatorname{Re} r \in [0, 1]$, (2) $|\operatorname{Re}(p-q)| < \operatorname{Re}(1-r)$; (3) the product $x^{1+p} f(x)$ is integrable in the whole segment $[0,a]$; if it has a singularity at the endpoint $x = 0$, then it is not higher than $O(x^{-\alpha})$, $0 < \alpha < 1$.

N o t e . By integration by parts (7.5) is reduced to a form which may be more convenient since the singularity at $x = a$ will be explicitly isolated

$$\varphi(x) = \frac{2^{1-r}x^{q+1}}{\Gamma(^1/_2(1+r+p+q))\,\Gamma(^1/_2(1+r+q-p))} \left\{ \frac{\psi(a)}{(a^2 - x^2)^{^1/_2(1-r+q-p)}} - \right.$$
$$\left. -\int_x^a \frac{\psi'(t)\, dt}{(t^2 - x^2)^{^1/_2(1-r+q-p)}} \right\} \tag{7.6}$$

where

$$\psi(t) = t^{-p-q-r} \frac{d}{dt}\int_0^t \frac{u^{1+p} f(u)\, du}{(t^2 - u^2)^{^1/_2(1-r+p-q)}}$$

P r o o f . Let us represent the functions F in (7.3) by the Euler integrals

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}\, dt$$

We have for the function in the first formula of (7.3)

$$F = \frac{\Gamma(1+p)}{\Gamma(^1/_2(1+r+q+p))\,\Gamma(^1/_2(1+p-q-r))} \times$$
$$\times \int_0^1 t^{^1/_2(-1+q+p+r)}(1-t)^{^1/_2(-1+p-q-r)}\left(1 - t\frac{x^2}{y^2}\right)^{^1/_2(-1+q-p-r)} dt \tag{7.7}$$

Because of the restrictions (1) and (2) imposed on the parameters, such a representation is admissible. Let us substitute $t = s^2/x^2$ into (7.7). Inserting the result into (7.3), we obtain

$$W_{p,q}^{(r)}(x, y) = \frac{2^{1+r}x^{-p}y^{-q}}{\Gamma(^1/_2(1-r+p-q))\,\Gamma(^1/_2(1-r+q-p))} \times$$
$$\times \int_0^x (x^2 - s^2)^{^1/_2(-1+p-q-r)}(y^2 - s^2)^{^1/_2(-1+q-p-r)} s^{p+q+r}\, ds \tag{7.8}$$

Performing the same calculation with the second formula of (7.3) we obtain an analogous result with the sole difference that the letters x and p have exchanged places with the letters y and q

Both results may be written as single Formula

$$W_{p,q}^{(r)}(x, y) = \frac{2^{1+r}x^{-p}y^{-q}}{\Gamma(^1/_2(1-r+p-q))\,\Gamma(^1/_2(1-r+q-p))} \times$$

$$\times \int_0^{\min(x,y)} \frac{t^{p+q+r}\,dt}{(x^2-t^2)^{^1/_2(1+r+q-p)}(y^2-t^2)^{^1/_2(1+r+p-q)}} \tag{7.9}$$

Let us introduce the notation (7.10)

$$\psi(y) = y^{-q}\varphi(y); \qquad g(x) = 2^{-1-r}\cdot\Gamma\left(\frac{1-r+p-q}{2}\right)\Gamma\left(\frac{1+r+q-p}{2}\right)x^p f(x)$$

Then Equation (7.4) becomes

$$\int_0^x \psi(y)\,dy \int_0^y \frac{t^{p+q+r}\,dt}{(x^2-t^2)^{^1/_2(1+r+q-p)}(y^2-t^2)^{^1/_2(1+r+p-q)}} +$$

$$+ \int_x^a \psi(y)\,dy \int_0^x \frac{t^{p+q+r}\,dt}{(x^2-t^2)^{^1/_2(1+r+q-p)}(y^2-t^2)^{^1/_2(1+r+p-q)}} = g(x) \tag{7.11}$$

In connection with the restrictions imposed on the parameters, a change in the order of integration is admissible in both components. Performing this change (by the Dirichlet formula in the first term) and combining the results, we reduce (7.4) to

$$\int_0^x \frac{t^{p+q+r}}{(x^2-t^2)^{^1/_2(1+r+q-p)}}\,dt \int_t^a \frac{\psi(y)\,dy}{(y^2-t^2)^{^1/_2(1+r+p-q)}} = g(x) \tag{7.12}$$

The problem has thereby been reduced to the double solution of Abel equations. Let us recall that the Abel equation

$$\int_0^x \frac{\varphi(t)\,dt}{(x^2-t^2)^m} = f(x)$$

with continuous right-hand side has the unique solution

$$\varphi(x) = \frac{2\sin m\pi}{\pi}\,\frac{d}{dx}\int_0^x \frac{yf(y)\,dy}{(x^2-y^2)^{1-m}}$$

in the case $0 < m < 1$ (under the condition that the integral converges at the lower limit). For $m < 0$ the solution exists only under additional restrictions on the right-hand side. Conditions (1), (2) and (3) guarantee unique solvability of Equation (7.12). Let us note that equivalence was not disturbed anywhere during the reduction of (7.4) to (7.12). Solving the Abel equations successively, we find

$$\omega(t) = \int_t^a \frac{\psi(y)\,dy}{(y^2-t^2)^{^1/_2(1+r+p-q)}} = t^{-p-q-r}\,\frac{2}{\pi}\cos\frac{(r+q-p)\pi}{2}\,\frac{d}{dt}\int_0^t \frac{ug(u)\,du}{(t^2-u^2)^{^1/_2(1-r+p-q)}} \tag{7.13}$$

$$\psi(x) = -\frac{2}{\pi}\cos\frac{(r+p-q)\pi}{2}\,\frac{d}{dx}\int_x^a \frac{t\omega(t)\,dt}{(t^2-x^2)^{^1/_2(1-r+q-p)}} \tag{7.14}$$

Combining these results and returning to the previous notations (7.10), we obtain (7.5) after simplifying the factor before the integral, q.e.d.

In conclusion, let us recall that the success of applying the apparatus of analytical theory of differential equations to (0.1) is due to the special form of the kernel (Section 1). This circumstance evidently should be taken into account in constructing mathematical models of a linearly deformable foundation.

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